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Two thin conducting, electrically neutral, parallel plates forming an isolated system in vacuum exert attracting force on each other, whose origin is the quantum electrodynamical interaction. This theoretical hypothesis, known as Casimir effect, has been also confirmed experimentally. Despite long history of the subject, no completely convincing theoretical analysis of this effect appears in the literature. Here we discuss the effect (for the scalar field) anew, on a revised physical and mathematical basis. Standard, but advanced methods of relativistic quantum theory are used. No anomalous features of the conventional approaches appear. The Casimir quantitative prediction for the force is shown to constitute the leading asymptotic term, for large separation of the plates, of the full, model-dependent expression.

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One of the significant visualizations of a quantized field in the early development of the relativistic quantum theory has been that of an ensemble of oscillators (see e.g. [1]). One found a close mathematical relation between a quantum field and an infinite collection of quantum harmonic oscillators, each characterized by its characteristic frequency ω_α, and taking on one of the energy values $E_{\alpha,n_\alpha} = \hbar\omega_\alpha(n_\alpha + \frac{1}{2})$. The collection of the amplitudes of the oscillators reflects the strength of the field. The lowest energy state of the field, the vacuum state, is represented by all of the oscillators being in their respective ground states. The mean value of the field vanishes in this state, but the mean value of the field strength *squared* does not. This shows, one says, that vacuum fluctuates, the energy of these fluctuations being equal to $(\hbar/2)\sum_\alpha \omega_\alpha$ (the "zero-modes sum", or "zero-point energy", as it has been christened). This sum is infinite, but, as one is told, the energy differences is what counts experimentally, not the absolute value.

This early picture has been, of course, superseded by later developments in the quantum field theory (QFT). In particular, one of the earliest instances of renormalization, normal ordering of operators, eliminated the infinite energy of the vacuum state and renormalized it to zero. Fluctuations of the field strength simply reflect the fact, that vacuum is not an eigenstate of the field (neither is any other state), but they are not dynamical, and cannot be a possible source of extractable energy. However, it is an astonishing fact, that there does exist an isolated area, in which the "zero-modes" ideology has been in use to date. In a 1948 paper [2] Casimir considered situation in which two thin conducting, electrically neutral, parallel plates are placed in the vacuum. The presence of these plates creates boundary conditions for the electromagnetic field, which change the characteristic frequencies of field oscillators to $\omega_\alpha(a)$, depending on the separation of the plates a. The lowest state is now characterized by the energy $(\hbar/2)\sum_\alpha \omega_\alpha(a)$. This is again infinite, but Casimir regularized it so as to squeeze a finite result out of it. Minus derivative of this expression is supposed to give the force between the plates. His prediction for the force, as is well known, was $-(\pi^2/240)\hbar c\, a^{-4}$. The paper addressed an interesting and fundamental quantum phenomenon, and is deservedly regarded as a pioneering work. The method it used, however, reflects the relatively early stage of the development of QFT, and today it should not be taken as a serious base of further research in this field, as it still very often is (see review articles on the subject [3]; the Casimir reasoning is even reported, virtually unchanged, by some respectable textbooks on modern QFT, see e.g. Ref. [4]). Other methods for calculating the Casimir energy have been developed [3], but they are not mathematically or physically sound either, as we shall argue below.

Consider the system investigated by Casimir, a quantum field plus plates, where for simplicity we take scalar field. Its most satisfactory description would be achieved, as usually in physics, by constructing a closed theory of both elements in mutual interaction. Here we leave this ambitious task aside, we would like to understand first the reasonable approximate idealization in which the plates are regarded infinitely heavy (hence, in particular, classical). We want to show, that even for that restricted purpose the usual naive treatment of the concept of quantum field is not sufficient. Proper care has to be taken with respect to the algebraic structure of the theory, including the scope of quantum variables under consideration, their various representations, and the time evolution of the system. The algebraic aspects of quantum theory, already stressed by Dirac in his monograph on quantum mechanics, has been growing in importance, especially with the creation of the Haag–Kastler framework for local algebras in quantum physics (see the monograph [5]). Today the algebraic approach is the most general and flexible framework for considering fundamental questions in quantum physics. We start our analysis by sketching the theory of free scalar field in the initial conditions formulation, stressing the algebraic aspects. We use the units with $\hbar = 1$, $c = 1$.

Let $\mathcal{L}_0$ denote the real vector space formed by pairs of functions on the 3-space $(v(\vec{x}), v_t(\vec{x}))$, each of which is an element of the vector space $\mathcal{D}_\mathbb{R}$ of real, infinitely differentiable functions with compact support ($\mathcal{L}_0$ is the direct sum $\mathcal{D}_{\mathbb{R}} \oplus \mathcal{D}_{\mathbb{R}}$). The elements of $\mathcal{L}_0$ will be denoted by $V \equiv (v, v_t)$, and the vector arguments suppressed. $\mathcal{L}_0$ becomes a symplectic space with the introduction of the symplectic form

$$\sigma(V_1, V_2) = \int (v_2(\vec{x}) v_{1t}(\vec{x}) - v_1(\vec{x}) v_{2t}(\vec{x}))\, d^3x\,. \tag{1}$$

The real scalar quantized field is a set of elements $\phi(V)$ generating an algebra by the relations

$$\phi(V)^* = \phi(V)\,,\ [\phi(V_1), \phi(V_2)] = i\sigma(V_1, V_2)\,. \tag{2}$$

(More precise formulation expresses the above relations in terms of Weyl elements $W(V) = \exp i\phi(V)$, in order to avoid domain problems.) The element $\phi((0, v_t))$ has the interpretation of the field operator"smeared" with the test function $v_t(\vec{x})$, and the element $\phi((v, 0))$ – the interpretation of the canonical momentum "smeared" with the test function $v(\vec{x})$; the elements are localized in the support of their test functions. The free, massless evolution of the quantum field is obtained by a simple "quantization" of the classical linear evolution determined by the wave equation. Denote by h_0 the square root of the positive operator $h_0^2 \equiv -\Delta$ acting on $L^2(\mathbb{R}^3, d^3x)$. Then the evolution of the initial conditions for this equation is given by the transformation

$$\begin{aligned} v_0(t) &= \cos(h_0 t)\, v + \sin(h_0 t)\, h_0^{-1}\, v_t\,, \\ v_{0t}(t) &= -\sin(h_0 t)\, h_0\, v + \cos(h_0 t)\, v_t \end{aligned} \tag{3}$$

(implying $v_0(0) = v$, $v_{0t}(0) = v_t$). Note that $v_0(t)$ and $v_{0t}(t)$ are in $\mathcal{D}_{\mathbb{R}}$, so $V_0(t) \equiv (v_0(t), v_{0t}(t)) \in \mathcal{L}_0$. The evolution is a symplectic transformation, that is,

$$\sigma(V_{10}(t), V_{20}(t)) = \sigma(V_1, V_2)\,. \tag{4}$$

The quantum field evolves according to automorphic map of the algebra

$$\alpha_{0t}\, \phi(V) = \phi(V_0(t))\,. \tag{5}$$

The next step is the construction of the vacuum representation – the unique representation in which the operator of energy has an eigenvector, and its spectrum is bounded from below. This is achieved, as is well known, by separating "positive frequencies" from "negative frequencies" in the evolution law of the field and interpreting the coefficients of this two parts as creation and annihilation operators respectively. More precisely, this amounts to the following. Take the Hilbert space $\mathcal{K} \equiv L^2(\mathbb{R}^3, d^3x)$ as the "one-particle space". By taking multiple direct products of $\mathcal{K}$, and then forming the direct sum of these products, construct a standard Fock space $\mathcal{H}$. Denote by Ω_0 the distinguished normalized vector in $\mathcal{H}$ ("Fock vacuum"), and by $a(f)$, $a^*(f)$, for $f \in \mathcal{K}$, the usual annihilation and creation operators in $\mathcal{H}$. The real-linear operator $j_0 : \mathcal{L}_0 \mapsto \mathcal{K}$ defined by

$$j_0(V) = 2^{-1/2} \left(h_0^{1/2}\, v - i h_0^{-1/2}\, v_t \right) \tag{6}$$

extracts the positive frequency part of the evolution:

$$j_0(V_0(t)) = e^{ih_0 t} j_0(V)\,, \tag{7}$$

(note that j_0 is well defined, as $\mathcal{D}_{\mathbb{R}} \subset \mathcal{D}(h_0^{1/2}) \cap \mathcal{D}(h_0^{-1/2})$). The vacuum representation $\pi_0\,(\phi(V)) \equiv \Phi_0(V)$ is now defined by

$$\Phi_0(V) = a(j_0(V)) + a^*(j_0(V))\,. \tag{8}$$

In this representation the evolution is implemented by the unitary operator $U_0(t) = \exp iH_0 t$, with H_0 having the interpretation of the field energy operator, by

$$U_0(t)\Phi_0(V)U_0(t)^{-1} = \Phi_0(V_0(t))\,. \tag{9}$$

If $\{f_i\}_{i=1}^{\infty}$ is any (orthonormal) basis of $\mathcal{K}$ in the domain of $h_0^{1/2}$, then H_0 may be represented by

$$H_0 = \sum_{i=1}^{\infty} a^*(h_0^{1/2} f_i)\, a(h_0^{1/2} f_i)\,. \tag{10}$$

Hence, in particular, H_0 is positive, and Ω_0 is the physical vacuum: $H_0\Omega_0 = 0$.

The theory is thus defined, but one should bear in mind three levels of specialization in the construction: Eqs. (1–2) define the algebra, Eqs. (3–5) the free evolution, and Eqs. (6–10) the vacuum representation. One should also point out, that the choice of the basic vector space for the canonical relations ($\mathcal{L}_0$ above) is to certain extent flexible, as long as all consistency conditions are satisfied as above.

Now we can return to our task of investigating Casimir effect. In the first step one has to define a one-parameter family of time evolutions of our field algebra, enforced by the presence of the conducting plates at all possible (but fixed) distances a (we place one of them in the x-y plane ($z = 0$), and another parallel at $z = a$). This should amount to imitating the steps embodied by Eqs. (3–5), with h_0 replaced by the square root h of the positive operator h^2 in $L^2(\mathbb{R}^3, d^3x)$ defined uniquely as $-\Delta$ with Dirichlet conditions on the plates. Here, however, one encounters a serious difficulty. The new classical evolution law, Eq. (3) with h replacing h_0, implies that v has to lie in $\mathcal{D}(h)$, the domain of h. Now, all functions in $\mathcal{D}(h)$ vanish on the plates ($\mathcal{D}(h)$ is equal to the direct sum of the three Sobolev spaces H_0^1 for each of the three closed regions into which the whole space is divided by the plates [6]). The new evolutions may not be defined on our algebra. Moreover, any other acceptable choice of the symplectic space will not solve the problem: with varying separation a one sees, that v would have to vanish in the whole region of interest, making the theory trivial. Physically this means, that the idealization of sharp Dirichlet conditions at variable positions is unphysical, at least in the approximation of heavy, classical plates. No traditional approach is able to clarify the source of this difficulty. Trying to ignore the difficulty, one is bound to encounter infinities of physical, and not only technical, origin.

The only possible solution is choosing some other model for the interaction with the plates, some "softened" version of the Dirichlet condition; this "softening" will affect the dynamics in the z-direction. Moreover, one should also expect difficulties coming from the infinite extension of the plates (they would not appear, e.g., for spherical shells). This is, however, not serious, as we are interested in quantities (e.g. force) per unit area of the plates, so we can approximate by large, but finite extension plates (taking limit at an appropriate point). What we propose, therefore, is the following. Put $h_\perp^2 = -\partial_x^2 - \partial_y^2$ on $L^2((-L_x/2, +L_x/2)\times(-L_y/2, +L_y/2), dx\, dy)$ with Dirichlet conditions at $x = \pm L_x/2$, $y = \pm L_y/2$, with large, but finite L_x, L_y; denote $h_{0z}^2 = -\partial_z^2$ on $L^2(\mathbb{R}, dz)$ and redefine $h_0^2 = h_\perp^2 + h_{0z}^2$. Change the model for plates by changing the operator of z-motion from h_z, which ensures strict Dirichlet condition, to $\tilde{h}_z$. For the moment we only assume that $\tilde{h}_z - h_{0z}$ is a bounded operator on $L^2(\mathbb{R}, dz)$, commuting with the complex conjugation. Finally, set $\tilde{h}^2 = h_\perp^2 + \tilde{h}_z^2$.

With this constructions the operators h_0^{-1} and $\tilde{h}^{-1}$ are bounded, whereas the domains of h_0 and $\tilde{h}$ are identical (the last statement follows from the equivalence of the norms on $\mathcal{D}(h_0)$: $\big(\|\psi\|^2 + \|h_0\psi\|^2\big)^{1/2}$ and $\big(\|\psi\|^2 + \|\tilde{h}\psi\|^2\big)^{1/2}$). We modify the choice of the field algebra by replacing the original space $\mathcal{L}_0$ by $\mathcal{L} = \mathcal{D}_\mathbb{R}(h_0) \oplus L^2_\mathbb{R}((-L_x/2, +L_x/2) \times (-L_y/2, +L_y/2) \times \mathbb{R}, d^3x)$, where the subscript $\mathbb{R}$ denotes the real part. The defining Eqs. (1,2) remain intact. It is now easy to show, that both the free h_0-evolution as well as all new $\tilde{h}$-evolutions are correctly defined on our new algebra by Eqs. (3–5) (for the $\tilde{h}$-evolutions obvious changes of notation are to be understood: $h_0 \to \tilde{h}$, $V_0(t) \to \tilde{V}(t)$, $\alpha_{0t} \to \tilde{\alpha}_t$). The $\tilde{h}$-evolutions are interpreted as the evolutions of the field under the external conditions created by the influence of plates.

The construction of the vacuum representation of the modified algebra remains unchanged, as outlined by Eqs. (6–10), except that now $\mathcal{K} = L^2((-L_x/2, +L_x/2) \times (-L_y/2, +L_y/2) \times \mathbb{R}, d^3x)$, and $j_0 : \mathcal{L} \mapsto \mathcal{K}$ in (6). By similar method one constructs "minimal energy state" representations with respect to each of the $\tilde{h}$-evolutions ("energy" means now the field energy together with interaction energy with fixed plates). The analog of Eq. (6) defines $\tilde{j}$ (well defined, as $\mathcal{D}(A) \subset \mathcal{D}(A^{1/2})$ for each positive A), and the analog of Eq. (7) shows its role. The new representations $\tilde{\pi}(\phi(V)) \equiv \tilde{\Phi}(V)$ are constructed in the same Fock space, and with the use of the same creation and annihilation operators, but with j_0 replaced by $\tilde{j}$ in the analog of (8). The $\tilde{h}$-evolution is implemented in this representation as in (9) if we replace $V_0(t)$ by $\tilde{V}(t)$, Φ_0 by $\tilde{\Phi}$ and $U_0(t)$ by $\tilde{U}(t) = \exp i\tilde{H}t$. $\tilde{H}$ is given by (10), with h_0 replaced by $\tilde{h}$, but only up to a multiple of the unit operator. This ambiguity becomes nontrivial if one changes the position of the plates (and, consequently, $\tilde{h}$), and is the result of our not having the full interacting theory at our disposal. The vector state Ω_0, however, with no ambiguity is the minimal $\tilde{H}$-energy state in this representation.

In the next step towards our goal one has to answer the question, whether various constructed representations are unitarily equivalent. If they are not, the situations to which they refer are physically noncomparable, and no quantities referring to the change of the distance between the plates may sensibly be determined. As the vacuum representation $\Phi_0(V)$ defines the energy of the field itself, we want to transform the other representations to this one. We ask therefore, whether there does exist for each $\tilde{h}$ a unitary operator Q such that $Q\,\tilde{\Phi}(V)\,Q^* = \Phi_0(V)$ for all $V \in \mathcal{L}$. To answer the question one uses standard methods. One can show that $j_0(\mathcal{L}) = \mathcal{D}(h_0^{1/2})$ and $\tilde{j}(\mathcal{L}) = \mathcal{D}(\tilde{h}^{1/2})$. Denote by K the operator of complex conjugation on $\mathcal{K}$ and define operators $T = 2^{-1}(j_0\tilde{j}^{-1} - ij_0\tilde{j}^{-1}i) = 2^{-1}(h_0^{1/2}\tilde{h}^{-1/2} + h_0^{-1/2}\tilde{h}^{1/2})$, $S = 2^{-1}(j_0\tilde{j}^{-1} + ij_0\tilde{j}^{-1}i) = 2^{-1}(h_0^{1/2}\tilde{h}^{-1/2} - h_0^{-1/2}\tilde{h}^{1/2})K$ transforming $\mathcal{D}(\tilde{h}^{1/2})$ into $\mathcal{D}(h_0^{1/2})$. The morphism $\Phi_0(V) \mapsto \tilde{\Phi}(V)$ may be equivalently expressed as a Bogoliubov transformation $a(f) \mapsto b(f) \equiv a(Tf) + a^*(Sf)$, for all $f \in \mathcal{D}(\tilde{h}^{1/2})$. This transformation is unitarily implementable, $b(f) = Q\,a(f)\,Q^*$, if, and only if, the operator S is a Hilbert–Schmidt operator [7], i.e. the trace of S^*S is finite (hence, in particular, T and S are bounded). When calculating this trace, one shows that the summation over the degrees of freedom parallel to the plates may be explicitly carried out, and for large dimensions of the plates (large L_x and L_y) one obtains

$$\frac{\operatorname{Tr} S^*S}{L_x L_y} \to \frac{\pi}{4}\operatorname{Tr}\,(\tilde{h}_z - h_{0z})^2\,, \tag{11}$$

where on the rhs. the operators and the trace are regarded as operations on $L^2(\mathbb{R}, dz)$. Thus to satisfy our requirements we assume that $\tilde{h}_z - h_{0z}$ is a Hilbert–Schmidt operator. We can describe, then, all situations of interest to the Casimir effect in the representation Φ_0. In particular, the state minimizing the sum of field energy and the energy of interaction with external conditions (the sum given by the operator $\tilde{H}$ in the representation $\tilde{\Phi}$), which was described by the vector Ω_0 in the representation $\tilde{\Phi}$, is given now by $\Omega = Q\Omega_0$.

Now we come to the determination of the Casimir force. In concord with the usual treatments we assume, that the states Ω (for varying position of the plates) transform adiabatically into each other. Contrary to implicit assumptions of most of the usual treatments, however, we think that for the purpose of calculating actual force one should compare the expectation value in these states of *the energy of the field itself* represented by the operator H_0, without including the interaction energy. We support this view by three arguments: (1) as pointed out above, H_0 is the only unambiguous energy operator in the problem, (2) in closed electrodynamics the interaction energy is absorbed by the pure (canonical) matter energy to form the full mechanical energy of the matter, (3) it is exactly the change in the classical analog of H_0 which is used for the calculation of the force exerted on a conductor in a classical electromagnetic field [8]. The quantity to be calculated is, therefore, $(\Omega, H_0\,\Omega) = (\Omega_0, Q^* H_0 Q\,\Omega_0)$. Using (10) and $Q^*\,a(f)\,Q = a(T^* f) - a^*(S^* f)$ one obtains $(\Omega, H_0\Omega) = 4^{-1}\mathrm{Tr}\,(\tilde{h} - h_0)\tilde{h}^{-1}(\tilde{h} - h_0)$. Summing the parallel degrees of freedom one gets for large L_x and L_y

$$\frac{(\Omega, H_0\Omega)}{L_x L_y} \to \frac{1}{24\pi}\,\mathrm{Tr}\,(\tilde{h}_z - h_{0z})(\tilde{h}_z + 2h_{0z})(\tilde{h}_z - h_{0z})\,. \tag{12}$$

If this is finite, the states Ω are energetically comparable, and the Casimir force may by determined.

Finally, we specify the "softened Dirichlet condition". We guess, that for the appearance of some universality in the Casimir effect, as incorporated by Casimir's original prediction, the behaviour of $\tilde{h}_z - h_{0z}$ at the lower end of the spectrum of both h_z and h_{0z} is decisive. We put, therefore $\tilde{h}_z - h_{0z} = f(h_z) - f(h_{0z})$, where f is a real smooth function on $\mathbb{R}_+$, with $f(u) = u$ for small u, $0 \leq f(u) \leq u$ for all u, and vanishing at least as u^{-2} for $u \to \infty$. This ensures finiteness of both (11) and (12). The resulting $\tilde{h}$ will not guarantee the relativistic causality of the evolution, but for the quasi-static idealization this is not a serious objection. Denote the Casimir energy (12) for this model by $\mathcal{E}(a)$, and introduce the abbreviation $\chi(\kappa, p) = \kappa^2(\kappa^2 - p^2)^{-2}(f(\kappa) - f(p))^2(3p - f(p) + f(\kappa))$. Then

$$\mathcal{E}(a) = \frac{1}{6\pi^3}\sum_{k=1}^{\infty}\frac{\pi}{a}\int_0^\infty dp\,\chi\Big(\frac{k\pi}{a}, p\Big)\,\big(1 + (-1)^{k+1}\cos ap\big) + \frac{1}{6\pi^3}\int_0^\infty d\kappa\int_0^\infty dp\,\chi\,(\kappa, p)\;. \tag{13}$$

A rather lengthy analysis of this expression shows that

$$\mathcal{E}(a) = \mathcal{E}(\infty) - \frac{\pi^2}{1440}\,a^{-3} + o(a^{-3})\,, \tag{14}$$

$$-\frac{d\mathcal{E}}{da}(a) = -\frac{\pi^2}{480}\,a^{-4} + o(a^{-4})\,. \tag{15}$$

One recognizes in these expressions the familiar Casimir terms – the leading asymptotic term in the force and the second leading term in the energy (which are one half of the corresponding terms for the electromagnetic Casimir quantities). They are here determined completely by the behaviour of the function f in the neighborhood of zero (in fact, the property $f'(0) = 1$ is all one needs). However, their meaning here is different, the force obeys this simple law only for sufficiently large separations of the plates. For every finite separation other (f-dependent) terms will dominate for f approaching identity. It becomes evident from (13), that reaching this limit is both physically and mathematically meaningless – the energy becomes infinite and the force indeterminate. Observe, also, that here the energy $\mathcal{E}(a)$ is always positive, as it should be. The physical interpretation of the f-dependent limit $\mathcal{E}(\infty)$ is the following: it equals twice the work which the external forces have to perform to create the configuration of the field surrounding a single plate (in this limit the plates and the configurations around them may be regarded as independent). One checks the consistency of this interpretation by repeating the calculation for the configuration with only one plate present in the whole space, and finding that the resulting energy is indeed one half of $\mathcal{E}(\infty)$.

The calculation of the effect as performed here used a class of models determined by the function f. However, as mentioned above, the leading term in the force comes from the spectral area where $\tilde{h}_z = h_z$, so it is probably more universal. At the same time Eq. (12) gives the method for the construction of other models, and corrections to the leading terms.

Lessons to be drawn for experimental verification of the Casimir effect are as follows: first, the universality is to be searched for at large separation of the plates, and second, for smaller separations model-dependent aspects take over.

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